UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-99**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

99-C1006-3-170

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (i) Let f be defined on an open interval containing the point a. Define what it means for f to be differentiable at a.

   (ii) If $f(x) = x^n$, n a positive integer, show that $f'(a) = na^{n-1}$.

   (iii) Let

$$f(x) = \begin{cases} x^3 + 8, & x \geq 2, \\ 12x - 8, & x \leq 2. \end{cases}$$

   Show that f is differentiable at 2 with derivative 12.

2. State and prove

   (i) Rolle's Theorem,

   (ii) the Mean Value Theorem.

   Suppose that f is defined and differentiable for every $x > 0$ and put $g(x) = f(x+1) - f(x)$. If $f'(x) \to 0$ as $x \to \infty$, prove that $g(x) \to 0$ as $x \to \infty$.

   If $f'(x) \to 1$ as $x \to \infty$, show that $f(x) \to \infty$ as $x \to \infty$.

3. (i) Use L'Hôpital's Rule to evaluate the following limits.

$$\lim_{x \to 0} \frac{\tan^3 x - \sin^3 x}{x^5},$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2},$$

$$\lim_{x \to 0} \frac{\log(1-x) + \log(1+x)}{x^2}.$$

   (ii) Suppose that f is differentiable in (a, b), and that $a < x < b$. Suppose also that $\alpha_n \to x$ and $\beta_n \to x$ as $n \to \infty$, where $x < \alpha_n < \beta_n < b$ for $n = 1, 2, 3 \ldots$. Show that the quotients

$$\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

   need not converge to $f'(x)$ as $n \to \infty$.

PLEASE TURN OVER

4. Suppose that f is defined on $[a, b]$ and that the first $(n+1)$ derivatives $f^{(1)}_{(x)}, \dots, f^{(n+1)}_{(x)}$ exist for all $x \in [a, b]$. Let the remainder term $R_{n,a}(x)$ be defined by

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n,a}(x),$$

for $a \leq x \leq b$.

Show that

$$R_{n,a}(x) = \frac{f^{(n+1)}(\xi)}{n+1!}(x-a)^{n+1}$$

for some $\xi \in (a, x)$.

If $f(x) = \cos x \sin x$, show that $R_{n,0}(x) \to 0$ as $n \to \infty$ and hence deduce that

$$\sin x \cos x = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{x^{2n+1}}{(2n+1)!}.$$

5. Let f be a bounded function on $[a, b]$, $a < b$.

Define the upper Riemann integral $\overline{\int_a^b} f(x)\,dx$ and the lower Riemann integral $\underline{\int_a^b} f(x)\,dx$, and show that

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

The function $f(x)$ is defined for x in $[0, 1]$ by setting $f(x) = 0$ if x is irrational and $f\left(\frac{p}{q}\right) = \frac{1}{q}$ if $p \geq 0$, $q > 0$ are integers with no common factors. Show that f is Riemann integrable on $[0, 1]$ and determine $\int_0^1 f(x)\,dx$.

6. (i) Suppose f is Riemann integrable on $[a, b]$ and $m \leq f(x) \leq M$, $\forall x \in [a, b]$. Suppose g is continuous on $[m, M]$ and write $h(x) = g(f(x))$, $\forall x \in [a, b]$. Show that h is Riemann integrable on $[a, b]$.

(ii) Show that the function $\cos\left[\frac{1}{x}\right]$, where $[x]$ denotes the integer part of x, is Riemann integrable on $[a, b]$, where $0 < a < b$.

CONTINUED

7. (i) Show that if f is a continuous function on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

(ii) Are the following functions uniformly continuous over the indicated interval? Give justifications.

(a) $\cos\left(\frac{1}{x}\right)$ over $\left(0, \frac{\pi}{2}\right)$,

(b) $\exp(-|x|)$ over $(-\infty, \infty)$,

(c) x^3 over $(0, 1)$.

END OF PAPER